Stepping Stones

Phil Daro
Progressions

1. Stepping stones within lesson
2. Across lessons, within unit
3. Through concepts across units and grades
Progressions are pathways of thinking

• Students are located along a progression.
• Like all of us, they move back and forth along the progression inside a single problem.
• Progressions map concepts: how concepts build on each other, depend on each other.
• Coherence and focus.
Making sense

A basic human response. We have to be trained to suppress it. Schools sometimes do this, especially in mathematics. Making sense is an interaction between prior knowledge and current experience: what is already known and a question.
Students’ Prior Knowledge

Students bring a variety of prior knowledge to each lesson...variety across students. This is a fundamental pedagogic challenge. The focus of my talk today.
Variety: the challenge

How do we bring students from their varied starting points to a common way of thinking, a common and precise use of language sufficient for an explanation of the mathematics to mean what it should to the student?
Variety: the Stepping Stones within a lesson

Where are the stepping stones from where students start to grade level mathematics?
Mile wide –inch deep
cause:
too little time per concept
cure:
more time per topic
= less topics
Why do students have to do math problems?

a) to get answers because Homeland Security needs them, pronto

b) I had to, why shouldn’t they?

c) so they will listen in class

d) to learn mathematics
Why give students problems to solve?

- To learn mathematics.
- Answers are part of the process, they are not the product.
- The product is the student’s mathematical knowledge and know-how.
- The ‘correctness’ of answers is also part of the process. Yes, an important part.
Answers are a black hole: hard to escape the pull

- Answer getting short circuits mathematics, making mathematical sense
- Very habituated in US teachers versus Japanese teachers
- Devised methods for slowing down, postponing answer getting
Answer getting vs. learning mathematics

• USA:

• How can I teach my kids to get the answer to this problem?
  
  Use mathematics they already know. Easy, reliable, works with bottom half, good for classroom management.

• Japanese:

• How can I use this problem to teach the mathematics of this unit?
Butterfly method
Less wide-more deep

People are realizing that Answer getting, as important as it truly is, is not the the goal. Making sense and making explanations of mathematics that make sense are the real goals. Learning tricks is superficial; understanding is deep...a solid foundation
Prior knowledge

There are no empty shelves in the brain waiting for new knowledge.

Learning something new ALWAYS involves changing something old.

You must change prior knowledge to learn new knowledge.
You must change a brain full of answers

• To a brain with questions.
• Change prior answers into new questions.
• The new knowledge answers these questions.
• Teaching begins by turning students’ prior knowledge into questions and then managing the productive struggle to find the answers
• Direct instruction comes after this struggle to clarify and refine the new knowledge.
Variety across students of prior knowledge is key to the solution, it is not the problem.
Getting to the mathematics

From variety of what students bring ... 
To common grade level content...ways of thinking with grade level mathematics.
Progression II.

• Not: covering a succession of topics
• Not:: below grade level means re-cover topics
• Yes: building knowledge, upgrading prior knowledge, always need more foundation work to build another storey
• Yes: within each problem- the whole progression
Unfinished Learning

• Long division example
• The whole progression is alive inside every problem, every lesson, every student
• Stay with the grade level problem and give more help: feedback and quaeetions
• Not: quit on the grade level and “reteach”
• How games motivate persistence and effort
Where are the stepping stones?

Students are standing on them
The variety of ways students think about a problem are the stepping stones to the grade level way of thinking.
Students explain their way of thinking...how they make sense of the problem, what confuses them, how they represent the problem, why the solution makes sense.
Students discuss how the different ways of thinking relate to each other.
Four Common Strategies for Differences among Students

1. Deny and Cover
2. Share and Wander
3. Differentiate and Forget about it
4. The Ways of Thinking are the Stepping Stones
Differences?

Fixed traits? Like “good at math/bad at math”
U.S. has a long tradition of “Remedies”,
and of snake oil.

Learning styles
Pace? Pace through what? The course? Ahead and behind.

Ways of thinking
Common humanity and differences

We all think
We all have prior knowledge
We all learn, i.e. revise prior knowledge
We all communicate
We are all different
Variety is wonderful
Variety is the foundation for learning, not the prob
...different ways of thinking relate...

Underlying mathematics often becomes visible, sensible, seeing it from different views
The different views come from differences in prior knowledge.
Stepping through different views and pulling them together by relating them, steps variety of students toward common understanding:
The grade level way of thinking; the target of the lesson.
the target of the lesson

Converge on the target
Explicit summary of the mathematics
Quote student work
Deny and cover

• Start lesson with lecture on grade level topic
• Show them how you want them to get the answer
• ignore evidence of variety such as student behavior, work and motivation, focus on compliance to procedure,
• “flunking” students is interpreted as high standards caused by non-compliance or misplacement
• It’s not my fault, what could I do
Where are the stepping stones?

The Ways of Thinking are the Stepping Stones.
The students are already standing on them

• Students’ ways of thinking *are* the stepping stones.
• Have a student closest to grade level way of thinking explain last (SAVE TIME)
• Have student with easy way of entering problem explain first
• Have two or three other ways of thinking explain in between, moving toward grade level
Upgrading prior knowledge

• By using variety as stepping stones, you are pulling all students toward grade level from wherever they start
• You are showing how new way of thinking relates to old
• You are upgrading prior knowledge
15 ÷ 3 = □
Show $15 \div 3 = \square$

1. As a multiplication problem
2. Equal groups of things
3. An array (rows and columns of dots)
4. Area model
5. In the multiplication table
6. Make up a word problem
Show $15 \div 3 = \square$

1. As a multiplication problem ($3 \times \square = 15$)
2. Equal groups of things: 3 groups of how many make 15?
3. An array (3 rows, $\square$ columns make 15?)
4. Area model: a rectangle has one side = 3 and an area of 15, what is the length of the other side?
5. In the multiplication table: find 15 in the 3 row
6. Make up a word problem
Show $16 \div 3 = \square$

1. As a multiplication problem
2. Equal groups of things
3. An array (rows and columns of dots)
4. Area model
5. In the multiplication table
6. Make up a word problem
Start apart, bring together to target

• Diagnostic: make differences visible; what are the differences in mathematics that different students bring to the problem
• All understand the thinking of each: from least to most mathematically mature
• Converge on grade-level mathematics: pull students together through the differences in their thinking
Next lesson

• Start all over again
• Each day brings its differences, they never go away
From Variety to Common Mathematical Understanding
Explain the mathematics when students are ready

• Toward the end of the lesson
• Prepare the 3-5 minute summary in advance,
• Spend the period getting the students ready,
• Get students talking about each other’s thinking,
• Quote student work during summary at lesson’s end
Start apart, bring together to target

- Diagnostic: make differences visible; what are the differences in mathematics that different students bring to the problem
- All understand the thinking of each: from least to most mathematically mature
- Converge on grade-level mathematics: pull students together through the differences in their thinking
Students Job: Explain your thinking

• Why (and how) it makes sense to you
  – (MP 1,2,4,8)

• What confuses you
  – (MP 1,2,3,4,5,6,7,8)

• Why you think it is true
  – (MP 3, 6, 7)

• How it relates to the thinking of others
  – (MP 1,2,3,6,8)
What questions do you ask

• When you really want to understand someone else’s way of thinking?
• Those are the questions that will work.
• The secret is to really want to understand their way of thinking.
• Model this interest in other’s thinking for students
• Being listened to is critical for learning
Students Explaining their reasoning develops academic language and their reasoning skills

Need to pull opinions and intuitions into the open:
  make reasoning explicit
Make reasoning public
Core task: prepare explanations the other students can understand
The more sophisticated your thinking, the more challenging it is to explain so others understand
Teach at the speed of learning

• Not faster
• More time per concept
• More time per problem
• More time per student talking
• = less problems per lesson
motivation

Mathematical practices develop character: the pluck and persistence needed to learn difficult content. We need a classroom culture that focuses on learning...a try, try again culture. We need a culture of patience while the children learn, not impatience for the right answer. Patience, not haste and hurry, is the character of mathematics and of learning.
Boise III

• Akihiko Takahashi, DePaul University

• Well designed, tested lessons grades 6-HS

• Progressions
  – http://ime.math.arizona.edu/progressions/

• https://www.illustrativemathematics.org/

• http://serpinstitute.org/

• http://collegeready.gatesfoundation.org/

• Insidemathematics.org
Video problem

• Convince
  – Yourself
  – A friend
  – A skeptic

That

\[2(n-1) = 2n - 2\]

Is true for any number, \(n\).
Four levels of learning

I. Understand well enough to explain to others
II. Good enough to learn the next related concepts
III. Can get the answers
IV. Noise
Four levels of learning

The truth is triage, but all can prosper

I. Understand well enough to explain to others
   As many as possible, at least 1/3

II. Good enough to learn the next related concepts
    Most of the rest

III. Can get the answers
    At least this much

IV. Noise
    Aim for zero
Efficiency of embedded peer tutoring is necessary
Four levels of learning
different students learn at levels within same topic

I. Understand well enough to explain to others
   An asset to the others, learn deeply by explaining

II. Good enough to learn the next related concepts
    Ready to keep the momentum moving forward, a help to others and helped by others

III. Can get the answers
    Profit from tutoring

IV. Noise
    Tutoring can minimize
When the content of the lesson is dependent on prior mathematics knowledge

• “I do – We do– You do” design breaks down for many students
• Because it ignores prior knowledge
• I – we – you designs are well suited for content that does not depend much on prior knowledge…
• You do- we do- I do- you do
Classroom culture:

• ....explain well enough so others can understand
• NOT answer so the teacher thinks you know
• Listening to other students and explaining to other students
Questions that prompt explanations

Most good discussion questions are applications of 3 basic math questions:

1. How does that make sense to you?
2. Why do you think that is true
3. How did you do it?
...so others can understand

- Prepare an explanation that others will understand
- Understand others’ ways of thinking
Minimum Variety of prior knowledge in every classroom; I - WE - YOU

Student A  Lesson START Level
Student B
Student C
Student D
Student E

CCSS Target Level
Variety of prior knowledge in every classroom; I - WE - YOU

Student A
Student B
Student C
Student D
Student E

Planned time
Needed time

Lesson START Level
CCSS Target Level
Variety of prior knowledge in every classroom; I - WE - YOU

Student A

Student B

Student C

Student D

Student E

Lesson START Level

CCSS Target Level
Variety of prior knowledge in every classroom; I - WE - YOU

Student A
Student B
Student C
Student D
Student E

CCSS Target
Answer-Getting
Lesson START Level
You - we – I designs better for content that depends on prior knowledge

Student A
Student B
Student C
Student D
Student E

Lesson START
Day 1
Lesson START
Level
Day 2
Attainment
Target
Differences among students

• The first response, in the classroom: make different ways of thinking students’ bring to the lesson visible to all

• Use 3 or 4 different ways of thinking that students bring as starting points for paths to grade level mathematics target

• All students travel all paths: robust, clarifying
What to look for

• Students are talking about each other’s thinking
• Students say second sentences
• Audience for student explanations: the other students.
• Cold calls, not hands, so all prepare to explain their thinking
• Student writing reflects student talk
Look for: Who participates

• EL students say second sentences
• African American males are encouraged to argue
• Girls are encouraged to engage in productive struggle
• Students listen to each other
• Cold calls, not hands, so no one shies away from mathematics
Shifts

1. From explaining to the teacher to convince her you are paying attention
   – To explaining so the others understand

2. From just answer getting
   – To the mathematics students need as a foundation for learning more mathematics

3. From variety of ways of thinking
   -- To grade level CCSS way of thinking: converge on the mathematics
Step out of the peculiar world that never worked

• This whole thing is a shift from a peculiar world that failed large numbers of students. We got used to something peculiar.

• To a world that is more normal, more like life outside the mathematics classroom, more like good teaching in other subjects.
A Whole in the Head

Fractions: Progression in the Common Core
Story 2

A lesson observed
counting

• Already an abstraction to count apples
• Count inches
• In base ten, we start counting tens
• In measurement we count $\frac{1}{2}$ inches
• We quarters of a dollar
Counting ones

- At some point, the number 3 can mean 3 ones....3 of 1. 30 can mean 3 of ten which is 10 of 1.
- Always starts with what 1 is.
- Word problems, rich problems: what does 1 mean in this situation?
Relating ones

• If there is time and distance, you have 1 minute and 1 mile...two ones. The relationship between the ones...how many times one is of the other ...gives the unit rate.

• Double number line, look at the 1s how many seconds at 1 mile? How many miles at 1 second?
Partitioning

- Partitioning starts in early grades with shapes,
- Very visual. Break a whole shape into equal parts.
- In third grade, partitioning is used to define fraction.
- Still Very visual.
Number line

Ruler in grades 1 and 2
Adding on the ruler in grade 2
Diagram of a ruler
Ask students to produce the diagram of a ruler and show their addition on their diagram.
What’s the difference between a diagram of a ruler and a diagram of a number line?
• What’s the difference between a diagram of a ruler and a diagram of a number line?
• At second grade, not much, except: the ruler shows inches and the number line shows numbers so it can be used to show anything you can count.
• At third grade: It shows numbers. A number is a point on the number line.
Red licorice

Made from organic cranberries.

Share among 4 friends: cut a length of red licorice into 4 equal lengths. How long is each piece?

If you cut the same length for 5 people, will the equal pieces be longer or shorter than the pieces for 4?

Students: Draw a diagram of red licorice and show why your answer makes sense so other students can understand you.
Partitioning one

Partition the length 1 into 4 equal parts. How long is a part?

On the number line, partition the length 0 to 1 into 4 equal parts. What is the number at the point where the first part from 0 ends? That point is the number $\frac{1}{4}$.

A fraction is a number, a point on the number line.
Of 1 we sing

• The length from 0 to $\frac{1}{4}$ is $\frac{1}{4}$ of 1 just as the length from 0 to 3 is 3 of 1.
• 1 is the “unit”.
• What we count in a situation = what 1 refers to in that situation
• What $\frac{1}{4}$ refers to is a number that is a part of 1.
Prior knowledge

• Students will have knowledge of partitioning a whole and a visual idea of a part. This is cognitively foundational, but not mathematically foundation.

• Work in 3rd and 4th grade should move students to partitioning 1 on the number line, more abstract and flexible. The unit 1 grows out of “whole, and replaces it. Out growing the whole takes work and thinking. What does 1 mean in this situation, what are we counting?
Future knowledge

Just as you count apples, inches and 10s, you can count $\frac{1}{4}$ s.

Slow down: 1, 2, 3 of $\frac{1}{4}$ of 1.

If you can count them, you can add them and subtract them. ..like apples or inches or tens.
Build on whole number arithmetic.

• If you can count them, you can use the same arithmetic you already know. The way you add and subtract whole numbers on the number line works exactly the same with unit fractions.

• I am adding quarters, I already know how.
Meanwhile...

• Go back to partitioning shapes (tape diagrams):
  • $2/4$ of a shape $= \frac{1}{2}$ of that shape
  • $2/4 = \frac{1}{2}$
• Grow forward to the number line, passing through red licorice as needed.
• Where are $2/4$ and $\frac{1}{2}$ on the number line?
2/4 and 1/4

• On the number line, 2/4 and ¼ are at the same place.
• They are the same point
• A point on the number line is a number
• Therefore they are the same number.
• Two different ways of writing the same number.
• I can replace one with the other in any calculation any time I want to. Whenever it serves my purpose.
Visuals help make sense

• See how multiplying the numerator and denominator of a fraction by the same number, $n$, corresponds physically to partitioning each unit fraction piece into $n$ smaller equal pieces.
Equivalent fractions

½ = 2/4 = 3/6 = 4/8 = 5/10 .......

3 = 3/1 = 6/2 ....

1 = 1/1 = 2/2 = 3/3 = 4/4 = 5/5 = ....

\[
\frac{a}{b} = \frac{an}{bn}
\]
I know how to generate equivalent fractions

• So I can change fractions to equivalent fractions that serve my purpose.

• I have a strip of red licorice 10 inches long. My little sister bit off 5 centimeters. How long is the remaining piece?
Focus in the standards

• No “simplify” fractions.
• Yes generate equivalent fractions that serve a useful purpose, often this goes in the direction of less simple.
• No term “improper fraction”
• Yes 5/3 is a fraction is a number is exactly like any other fraction.
Need common units

• Change 5 centimeters into the equivalent length in inches. Common units, common denominator, common denominator.

• Add \( \frac{3}{4} \) to \( \frac{5}{6} \)

• Need common units.

• What are the units? The unit fractions.
\[ \frac{3}{4} + \frac{5}{6} = \]
$\frac{18}{24} + \frac{20}{24} = \frac{38}{24}$

$\frac{3 \times 6}{4 \times 6} + \frac{5 \times 4}{6 \times 4} = \frac{38}{24}$
Look ahead...

<table>
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<th></th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
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<th>24</th>
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<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>32</th>
<th>36</th>
</tr>
</thead>
</table>

Ratio table
In whose head is the whole?
Dancing partitions
Where is \( \frac{1}{6} \)?

\[ \frac{1}{4} \]
Since 4 of $\frac{1}{4}$ in one, and 6 little parts in $\frac{1}{4}$, then $4 \times 6 = 24$ little parts in one.

\[ \overline{\phantom{000000000000000000000}} \]

Count them 4
\[ \frac{1}{6} \times \frac{1}{4} \]

Because \( \frac{1}{6} \times \frac{6}{1} = \frac{1}{4} \) of 1

\[ \frac{1}{4} \]
Count them.

4 groups of 6 each = 24

\[ \frac{1}{4} \times 24 = 6 \text{ left} \]
How many Little guys?

\[
\frac{3 \times 6}{4 \times 6} = \frac{18}{24} \quad \frac{5 \times 4}{6 \times 4} = \frac{20}{24} \\
(18 + 20) \frac{1}{24} = \frac{318}{24} \quad \frac{1}{24} + \frac{1}{24} = \frac{1}{4}
\]
Show that the length of a part from cutting 1 into 4 parts and then that $\frac{1}{4}$ length into 6 equals...
equals
the length cutting
1 into 6 parts
and then that
into 4 parts.
Jaime has to travel from his home on one side of a circular lake to the store on the opposite side. He has his choice of canoeing to the other side of the lake at a speed of 4 mph or running on a trail along the bank of the lake at a speed of 7 mph. If the lake is 2 miles across, what method would be the most efficient, and why?